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**LETTER TO THE EDITOR**

# Integrable systems admitting topological solitons

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**Abstract.** We discuss (1 + 1)-dimensional sigma models and Heisenberg models in which the target space has the topology of a cylinder. In these integrable systems, the solutions are classified by a winding number.

## 1. Introduction

This letter deals with integrable (1 + 1)-dimensional systems, in which the configuration space or phase space has disconnected sectors classified topologically by an integer (a winding number). The general set-up involves a field $\Phi(x,t)$ taking values in some manifold $M$. Here $t \in \mathbb{R}$ denotes time, and $x \in X$ is the space variable; either $X = \mathbb{R}$ and we impose a boundary condition $\Phi(-\infty,t) = \Phi(\infty,t)$, or $X = S^1$ (i.e. $\Phi$ is periodic in $x$). Furthermore, $M$ is chosen so that its fundamental group equals the group of integers $\mathbb{Z}$. So for each fixed $t$, $\Phi(\cdot,t)$ is in effect a continuous mapping from a circle into $M$, and it therefore has a winding number $N \in \mathbb{Z}$. As $t$ changes, this integer remains constant. One could also view a solution $\Phi$ as wrapping the spacetime $X \times \mathbb{R}$ around the image manifold $M$, with this mapping having winding number $N$ (in an appropriate sense).

The prototype in the $X = \mathbb{R}$ case is the sine-Gordon equation, where $M = S^1$. If we think of $\Phi$ as a two-dimensional unit vector ($\Phi \cdot \Phi = 1$), then the sine-Gordon system corresponds to the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\Phi_\mu \cdot \Phi_\nu - (1 - K \cdot \Phi) \tag{1}$$

where $K$ is a constant unit vector. Hereafter $x^\mu = (x^0, x^1) = (t, x)$ are the spacetime coordinates; a subscript denotes partial differentiation; and $\eta^{\mu\nu} = \mathrm{diag}(1,-1)$ is the (inverse) spacetime metric. An $N$-kink solution of the sine-Gordon equation has winding number $N$.

In our examples $M$ will be the cylinder $S^1 \times \mathbb{R}$, which (at least to begin with) we think of as the hyperboloid of one sheet in $\mathbb{R}^3$. One can visualize the field as a closed string which is wound around this cylinder, and evolves in time. Instead of $\Phi$, let us use the symbol $\psi^a$, denoting a three-dimensional vector satisfying

$$\eta_{ab}\psi^a\psi^b = 1 \tag{2}$$

where $\eta_{ab} = \mathrm{diag}(1,1,-1)$. The metric on the hyperboloid (2) is taken to be the one induced by the metric $\eta_{ab}$; $M$ is then a symmetric space $SO(2,1)/SO(1,1)$. Two naturally defined systems taking values on $M$ are the nonlinear sigma model, and the Heisenberg model (Landau–Lifshitz equation). The $\sigma$-model is defined by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}\psi^a_\mu\psi^b_\nu\eta_{ab} \tag{3}$$

while the Landau–Lifshitz equations arise from the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}\psi_x^a \psi_x^b \eta_{ab} \tag{4}$$

with Poisson brackets

$$\{\psi^a(x), \psi^b(y)\} = -\delta(x-y)\varepsilon^{abc}\psi^d(x)\eta_{cd}. \tag{5}$$

The analogous $S^2$ systems, in which $\eta_{ab}$ is replaced by the Euclidean metric $\delta_{ab}$, are of course well known integrable systems; and their hyperbolic versions are therefore integrable too, simply by analytic continuation. (By integrability we mean the existence of a suitable Lax pair.) In fact, these hyperbolic versions have been widely studied in their own right: cf [1–6]. The aim in what follows is to examine the simplest 'winding' solutions of the systems (3) and (4), (5).

## 2. The hyperbolic Heisenberg model

The Landau–Lifshitz equation obtained from (4) and (5) is

$$\psi_t^a = \eta^{ab}\varepsilon_{bcd}\psi^c\psi_{xx}^d. \tag{6}$$

If we parametrize the hyperboloid in terms of 'polar angles' as

$$\psi^a = (\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta)$$

then (6) is equivalent to

$$\theta_t = 2(\sinh\theta)\theta_x\phi_x + (\cosh\theta)\phi_{xx} \tag{7}$$

$$\phi_t = (\operatorname{sech}\theta)\theta_{xx} + (\sinh\theta)(\phi_x)^2 \tag{8}$$

while if we parametrize in terms of a stereographic projection as

$$\psi^a = \frac{1}{1+u^2-v^2}(1-u^2+v^2, 2u, 2v)$$

then (6) becomes

$$u_t = -v_{xx} - 2\alpha v(u_x^2+v_x^2) + 4\alpha u u_x v_x \tag{9}$$

$$v_t = -u_{xx} + 2\alpha u(u_x^2+v_x^2) - 4\alpha v u_x v_x \tag{10}$$

where $\alpha = (1+u^2-v^2)^{-1}$.

Let us first look for static winding solutions, by solving (7), (8) with $\theta$ and $\phi$ being functions of $x$ only. From (7) we obtain

$$\phi_x = B\operatorname{sech}^2\theta \tag{11}$$

where $B$ is a constant; and then (8) integrates to

$$\theta_x = \sqrt{A + B^2\operatorname{sech}^2\theta} \tag{12}$$

with $A$ constant. For a winding solution, we want $\theta(x)$ to be non-monotonic, which requires $-B^2 \leqslant A < 0$. Write $A = -N^2$, where $0 < N \leqslant B$. Then the solution of (12) is

$$\theta(x) = \sinh^{-1}\left(\sqrt{B^2/N^2 - 1}\,\sin(Nx)\right) \tag{13}$$

and this is periodic, with period $2\pi$, provided $N$ is an integer. Finally, $\phi(x)$ is obtained by integrating the smooth function (11). To check that we have a winding solution, it is sufficient to compute

$$\Delta\phi = \int_0^{2\pi} B\operatorname{sech}^2\theta\,\mathrm{d}x \tag{14}$$

we find that $\Delta\phi = 2\pi N$, and so we have a space-periodic static solution, with winding number $N$.

Note that if $B = N$ in (13), then $\theta = 0$ and $\phi = Nx$: a solution which winds $N$ times around the 'waist' of the hyperboloid. The simplest time-dependent solution is a generalization of this, namely $\theta \equiv$ constant, $\phi = Nx + N^2(\sinh\theta)t$.

For these solutions, the space $X$ is the circle; in fact, there are no travelling-wave winding solutions for $X = \mathbb{R}$. One can find a simple time-dependent solution on $\mathbb{R}$ by using the stereographic form (9), (10) of the equations. Let us look for solutions in which $\alpha$ is a function of $x$ only, i.e. take $u^2 - v^2 = f(x)^2$. It follows from (9) and (10) that the function $g = vu_x - uv_x$ satisfies

$$\frac{\partial g}{\partial x} = 2g\frac{\partial}{\partial x}\log(1 + f^2).$$

If we take the simplest solution of this, namely $g \equiv 0$, then $u$ and $v$ must have the form

$$u(x,t) = f(x)\cosh h(t)$$
$$v(x,t) = f(x)\sinh h(t).$$

Substituting these into (9), (10) gives $\mathrm{d}h/\mathrm{d}t = -m$ constant, and

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{2f}{(1+f^2)}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2 + mf. \tag{15}$$

This has the first integral

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2 = c(1+f^2)^2 - m(1+f^2)$$

where $c$ is an arbitrary constant. For winding solutions one needs $c \geqslant 0$. If $c = 0$, one obtains a solution which is equivalent to (18) below; so take $c > 0$, and scale $x$ so that $c = 1$. Now the requirement of winding imposes a restriction on $m$, namely $m \leqslant 1$. For $m < 0$ one obtains solutions which are similar to those for $m \geqslant 0$, so let us take $0 \leqslant m \leqslant 1$. Then the solution of (15) is

$$f(x) = \sqrt{1-m}\,\mathrm{sc}(x|m) \tag{16}$$

in the elliptic-function notation of [7]. The two limits of (16) are

$$f(x) = \tan x \qquad \text{for } m = 0 \tag{17}$$

and (after a shift in $x$)

$$f(x) = \mathrm{cosech}\, x \qquad \text{for } m = 1. \tag{18}$$

Therefore, we have a family of winding solutions parametrized by $m \in [0, 1]$, with $h(t) = -mt$ and where $f(x)$ is specified in (16), (17) or (18). For (16) and (17), the solution is periodic (i.e. $X = S^1$); while for (18) it lives on $X = \mathbb{R}$.

A Lax pair corresponding to the equation (6) is as follows. Define a $2 \times 2$ matrix $S \in SL(2,\mathbb{R})$ by

$$S = \begin{bmatrix} \psi^1 & \psi^2 + \psi^3 \\ \psi^2 - \psi^3 & -\psi^1 \end{bmatrix}.$$

Then the consistency condition for the linear system

$$\Psi_x = \lambda S\Psi$$
$$\Psi_t = -\lambda(2\lambda S + S_x S)\Psi$$

is exactly (6). Implementation of the inverse scattering transform (on $X = \mathbb{R}$) produces $N$-soliton solutions, and these turn out to have winding number $N$. The solution corresponding to (18), which in terms of $\psi^a$ is

$$\psi^a = (1 - 2\,\mathrm{sech}^2\, x, 2\,\mathrm{sech}^2\, x \sinh x \cosh t, -2\,\mathrm{sech}^2\, x \sinh x \sinh t)$$

is the simplest example: a stationary 1-soliton with winding number $N = 1$. These solitons are closely related to those of the 'standard' Heisenberg model [8].

## 3. The hyperbolic sigma model

In terms of the $(\theta, \phi)$ parametrization, the $\sigma$-model equations are

$$\theta_{tt} - \theta_{xx} = -\cosh\theta \sinh\theta(\phi_t^2 - \phi_x^2) \tag{19}$$

$$(\phi_t \cosh^2\theta)_t = (\phi_x \cosh^2\theta)_x. \tag{20}$$

For the static problem, the equations (and hence their solutions) are the same as for the static Heisenberg case of the previous section. What follows are some examples of time-dependent winding solutions.

First, let us look for solutions on $X = S^1$ for which $\phi = x$. (To go to winding number $N > 1$ is an easy generalization.) It follows from (19), (20) that $\theta$ is a function of $t$ only, satisfying

$$\theta_t^2 = c + \sinh^2\theta$$

where $c$ is an arbitrary constant. This is easily integrated in terms of elliptic functions. Apart from the case $c = 0$, the solutions have the property that $\theta(t)$ reaches infinity in finite time. An example is $c = 1$, where the solution

$$\theta(t) = \sinh^{-1}\tan t$$

goes from $\theta = -\infty$ to $+\infty$ as $t$ goes from $-\pi/2$ to $+\pi/2$. For $c > 0$ the solutions all have this behaviour, whereas for $c < 0$, $\theta$ comes in from infinity, turns round, and goes out again. In the limiting case $c = 0$, the solution

$$\theta(t) = 2\tanh^{-1}(k\mathrm{e}^t)$$

includes the static case $\theta \equiv 0$, and tends asymptotically to $\theta = 0$ as $t \longrightarrow -\infty$.

Our second family of solutions arises from the 'self-duality' equations

$$\phi_x = (\mathrm{sech}\,\theta)\theta_t$$

$$\phi_t = (\mathrm{sech}\,\theta)\theta_x$$

which imply (19), (20). So $\phi$ and $\mu = 2\tan^{-1}\exp\theta$ satisfy $\phi_x = \mu_t, \phi_t = \mu_x$, and are therefore 'conjugate' solutions of the $(1+1)$-dimensional wave equation. The general solution is

$$\phi = f(x+t) + g(x-t)$$

$$\mu = f(x+t) - g(x-t)$$

where $f$ and $g$ are arbitrary functions. All winding solutions are defined only on a finite time interval. For example, the choice

$$f(\xi) - \frac{\pi}{4} = g(\xi) + \frac{\pi}{4} = \frac{1}{2}\pi\tanh\xi$$

leads to

$$\phi = \frac{\pi \sinh 2x}{\cosh 2x + \cosh 2t}$$
$$\mu = \frac{\pi}{2} + \frac{\pi \sinh 2t}{\cosh 2x + \cosh 2t}$$

which is a winding solution on $\mathbb{R}$ (with unit winding number). However, bearing in mind that we need $0 < \mu < \pi$, we see that it represents a smooth solution only for $|t| < \frac{1}{2}\log 3$.

## 4. Positive-definite versions

In the systems discussed above, the metric on the target space $M$ is indefinite:

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{ab}\,\mathrm{d}\psi^a\,\mathrm{d}\psi^b \\ &= -\mathrm{d}\theta^2 + \cosh^2\theta\,\mathrm{d}\phi^2. \end{aligned}$$

One can replace this by an analogous positive-definite metric on the cylinder, namely

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \cosh^2\theta\,\mathrm{d}\phi^2. \tag{21}$$

The corresponding equations remain integrable, since they are obtained by simply making the replacement $\phi \mapsto \mathrm{i}\phi$ (and, for the Landau–Lifshitz case, also $t \mapsto -\mathrm{i}t$). We are still thinking of $\phi$ as being a periodic coordinate (and looking for solutions which wind in $\phi$); as a consequence of this, $M$ is no longer a symmetric space.

Let us briefly look at the corresponding 'sigma model'. From the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\eta^{\mu\nu}[(\cosh^2\theta)\phi_\mu\phi_\nu + \theta_\mu\theta_\nu]$$

one obtains the equations of motion

$$\theta_{tt} - \theta_{xx} = \cosh\theta\sinh\theta(\phi_t^2 - \phi_x^2)$$
$$(\phi_t\cosh^2\theta)_t = (\phi_x\cosh^2\theta)_x.$$

The most general static winding solution is now $\theta \equiv 0, \phi = Nx$. This is what one would expect: in the positive-definite case, the string will try to minimize its length, and $\theta = 0$ is where the cylinder is narrowest (with respect to the metric (21)). If we look for more general solutions having $\phi = Nx$, then $\theta$ has to be a function of $t$ only, with

$$\tanh\theta = \sqrt{m}\,\mathrm{sn}(\rho t|m)$$

where $\rho$ and $m$ are constants with $\rho > |N|$ and $m = 1 - N^2/\rho^2$. In other words, the string oscillates between the values $\theta_\pm = \pm\tanh^{-1}\sqrt{m}$.

## 5. Concluding remarks

There are several examples of integrable elliptic systems of partial differential equations admitting topological soliton solutions: for example, instantons in sigma models on $\mathbb{R}^2$ and gauge theory on $\mathbb{R}^4$, and BPS monopoles on $\mathbb{R}^3$. In these cases, there is no time dependence.

Analogous time-dependent examples (in other words, hyperbolic or parabolic systems, rather than elliptic) are not as prevelant: in fact, sine-Gordon is the only well known example. But many such integrable systems exist, and in this note we have briefly examined a few of them. It might be of interest to study further the inverse scattering transforms for these cases, and to try to attempt a general classification of systems of this type.

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